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A Family of $N = 2$ Gauge Theories with Exact S-Duality

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ABSTRACT

We study an infinite family of $N = 2$ $Sp(2n)$ gauge theories that naturally arise from the D3-brane probe dynamics in F-theory. The matter sector consists of four fundamental and one antisymmetric tensor hyper multiplets. We propose that, in the limit of vanishing bare masses, the theory has exact $SO(8) \ltimes SL(2, Z)$ duality. We examine the semiclassical BPS spectrum in the Coulomb phase by quantizing various monopole moduli space dynamics, and show that it is indeed consistent with the exact S-duality.

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1 Introduction

Electromagnetic duality is a powerful idea that relates strongly coupled theories to weakly coupled ones. Montonen and Olive's conjecture of the duality has been shown to be realized very simply in the $N = 4$ supersymmetric Yang-Mills theory [1]. This theory is finite and so the duality is expected to be exact in all energy scales. Sen was the first to realize that the larger $SL(2, Z)$ duality holds in this theory with gauge group $SU(2)$ [2]. Such S-duality has been also studied with higher rank groups. Furthermore, Seiberg and Witten proposed that a $N = 2$ supersymmetric Yang-Mills theory with $SU(2)$ gauge group and four massless flavors of matter fields in the fundamental representation, also has the exact S-duality in the Coulomb phase [3]. This conjecture was further supported from the study of the low energy quantum mechanics of dyons of magnetic charge one and two [4] and from the study of effective action [5].

Since then, there have been attempts to find to other $N = 2$ models beyond $SU(2)$ gauge group (such as $SU(3)$) with an exact S-duality, none of which succeeded [6, 7]. It has been also suggested that maybe the S-duality of the $SU(2)$ case is accidental and the scale-invariant $N = 2$ models need not be S-dual in general. In this work we study an infinite family of scale-invariant $N = 2$ supersymmetric Yang-Mills theory, and propose that they all possess exact S-duality in much the same way as the scale-invariant Seiberg-Witten $SU(2)$ does. The proposed theory has the gauge group $Sp(2n)$ with a single antisymmetric tensor and four fundamental matter fields. We will provide substantial evidences supporting the exact S-duality by studying the multi-monopole dynamics and the resulting semiclassical spectrum. Seiberg and Witten's $SU(2) = SP(2)$ model is clearly the first of the series of our model since the antisymmetric tensor of $Sp(2n)$ is a singlet and decouples for $n = 1$.

This particular generalization is in fact well-motivated by considering the simplest compactification of F-theory [8, 9, 10, 11]. In the simplest reincarnation of F-theory, one is effectively compactifying a type IIB theory on a two sphere but in such a manner that the dilaton-axion field varies with the holomorphic coordinate on the sphere. The consistency requires 24 singular points on the sphere, which extend to the remaining 7 spatial dimensions and can be thought of as a kind of D7-branes. Sen observed [9] that, when the sphere gets large and 24 singular points are split

into four groups of 6 singularities, each congregating together far away from the others, the local geometry of the two sphere near each group is identical to the vacuum moduli space of a $N = 2$ $SU(2)$ model with four (massive) fundamental hyper multiplets, the exact solution of which was found by Seiberg and Witten some time before. This remarkable fact was quickly explained by the observation that the $SU(2)$ theory in question is simply a world-volume Yang-Mills theory on a probe D3-brane parallel to D7-branes [10].

The limit of massless hyper multiplets is achieved by letting each group of the six nearby singularities to coincide at four different points. In this limit, the string background being probed by the D3-brane is rather trivial. There is no Ramond-Ramond charge that may transform nontrivially under the $SL(2, Z)$ U-duality of the type IIB string, and the dilaton-axion becomes uniform. As a matter of fact, the theory is an orientifold of type IIB theory with all 16 D7-branes localized at the four orientifold planes, and is dual to a good old orbifold T^2/Z_2 compactification of the type I string. Since this string background is invariant under the $SL(2, Z)$ U-duality of the type IIB string up to a trivial redefinition of the coupling, one could argue that the D3-probe dynamics must be similarly invariant under certain S-duality.

Recently, by generalizing the above idea to include the n D3-brane probes, two independent groups studied a series of $N = 2$ world-volume theory on the D3-branes [12]. The gauge group is $Sp(2n)$, and the matter multiplet includes four fundamental and a single antisymmetric tensor. Again, in the limit where quartets of D7-branes coincide with the orientifold plane, one would expect to recover a theory with an exact S-duality. This provides an excellent motivation for studying the proposed generalization of the scale-invariant Seiberg-Witten model.

This F-theory consideration suggests the whole family of $Sp(2n)$ models share the same duality group. With $n = 1$, the duality group is known to be $SO(8) \ltimes SL(2, Z)$. The $SO(8)$ is the global symmetry of the theory for all n, and rotates the fundamental hyper multiplets and their charge conjugates. This $SO(8)$ acts on neither the adjoint nor the antisymmetric fields. As in the $n = 1$ case previously studied, we will find that the $SL(2, Z)$ action involve a permutations of the vector 8_v, the spinor 8_s and the chiral 8_c representations of $SO(8)$. When the fundamental matters disappear for example by becoming infinitely massive, the duality group, which now has a new interpretation as a monodromy group, collapses to $\Gamma(2)$. The nontrivial triality action of $SL(2, Z)$

is naturally encoded in the fact that the six coset elements $SL(2,Z)/\Gamma(2)$ form a permutation group of $(8_v, 8_s, 8_c)$.

Our goal here is to confirm that the semiclassical BPS spectrum is indeed invariant under such S-duality, by quantizing various multi-monopole dynamics. One complication of having a larger gauge group is that the Coulomb phase structure is rather complicated even after assuming the maximal breaking to $U(1)^n$. There are two adjoint Higgs fields in $N = 2$ vector multiplet, and their vacuum expectation values must commute with each other. But when the gauge group has rank larger than one, the two Higgs expectations need not be parallel within the Cartan subalgebra [7, 13]. Depending on whether they are aligned or not, the multi-monopole dynamics will differ drastically. The 4 new Higgs from the antisymmetric hyper multiplets can make matters even more complicated.

When two vacuum values of the Higgs fields are not parallel, a monopole corresponding to each and every root is fundamental in the sense that there are only four bosonic zero modes. In order to test the self-duality of the BPS spectra, it is sufficient to consider the interactions of identical monopoles. By the fermion zero mode analysis we show that in this case the degeneracy and the dyonic excitations of a single monopole match exactly the prediction of the duality. In fact, more generally, the problem of counting states can be mapped either to the $N = 2$ $SU(2)$ case studied by Gauntlett and Harvey as well as Sethi, Stern and Zaslow [4] or (very surprisingly) to a $N = 4$ $SU(2)$ case studied by Sen and subsequently by others [2, 14]. In this way, we will find that the S-duality of the scale-invariant Seiberg-Witten $SU(2)$ model combined with the S-duality of the $N = 4$ $SU(2)$ guarantees that the same is true for the $N = 2$ $Sp(2n)$ theories for all n. As will be clear later on, such a reduction of the monopole dynamics to other known problems is the prevailing theme throughout this work.

In such generic vacua, the problem of finding BPS spectrum can be carried out from the F-theory viewpoint as in Ref. [11]. However, this method is ineffective in probing threshold bound states that necessarily arise when the vacuum expectation values are aligned; Because $N = 2$ theories are known to have discontinuities in the dyon spectrum, one must check for the dyon spectrum explicitly in all cases [3].

When the two expectation values are parallel, the magnetic monopole moduli space becomes quite rich. We can choose n fundamental monopoles, from which a configuration of any magnetic charge can be built [15]. The bosonic moduli space of arbitrary number of distinct monopoles has been found [16, 17]. Studying the index bundles on such moduli space, we can again relate many of the bound state problems to those found in $N=4$ Yang-Mills theory [18, 19]. Similarly as above, at least a substantial part of the S-dual BPS spectrum can be shown to arise from the seemingly unrelated fact that $N=4$ Yang-Mills theory has the exact S-duality of $SL(2,Z)$.

The plan of this work is as follows. In Section 2, we briefly review properties of the $SP(2n)$ gauge algebra and the elementary matter content. For simplicity, we shall set the antisymmetric Higgs expectation to zero without a loss of generality. In Section 3, we consider the magnetic monopole spectrum when the vacuum expectation values of two Higgs fields are not aligned. In Section 4, we consider special cases when the expectation values are parallel. The moduli space dynamics are far more subtle here, but turned out be to tractable in many cases. We review some basic fact about the index bundle spanned by the fermionic zero modes, which will help us to relate the present bound state problems to well-known ones in some $N=4$ theories. We conclude in Section 5 with some remarks.

2 The $Sp(2n)$ Lie Algebra

In this section, we set up the notation and describe the matter content briefly. The roots and the weights of $Sp(2n)$ algebra are easily described by n orthonormal vectors e_i in R^n such that $e_i \cdot e_j = \delta_{ij}$. With the Cartan generators H_i, $i=1,\ldots,n$ normalized by

$$\mathrm{tr}\, H_i H_j = \delta_{ij}, \tag{1}$$

the $2n^2$ roots of $Sp(2n)$, or equivalently the weights of the adjoint representation, are given by $\{\alpha\} = \{\gamma_i, \beta^{\pm}_{ij}, -\gamma_i, -\beta^{\pm}_{ij}\}$ where we have defined

$$\gamma_i = \sqrt{2}\, e_i, \tag{2}$$

$$\beta^{\pm}_{ij} = \frac{1}{\sqrt{2}}\,(e_i \pm e_j) \quad i > j. \tag{3}$$

Furthermore, the subset $\{\beta^{\pm}_{ij}, -\beta^{\pm}_{ij}\}$ contains all $2n^2 - 2n$ nonzero weights of an antisymmetric tensor, while the $2n$-dimensional fundamental representation has the weights $\{\gamma_i/2, -\gamma_i/2\}$. For all three representations, the multiplicity is one for any nonzero weight.

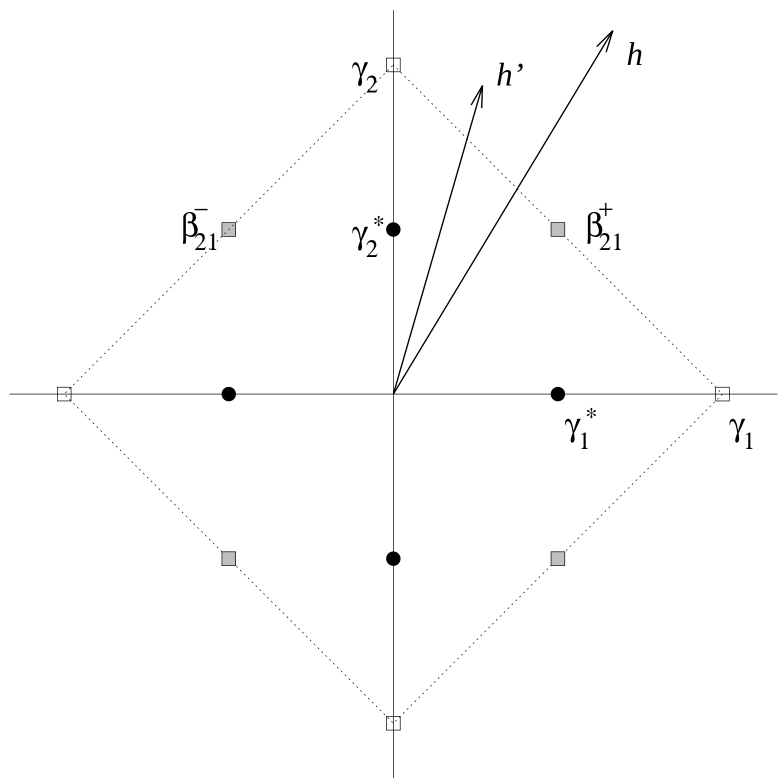

Figure 1: A diagram that shows the relevant weights of $Sp(4)$ gauge algebra. The eight roots are found along the dotted line.

The gauge symmetry is spontaneously broken to $U(1)^n$ for adjoint generic Higgs values, and in the unitary gauge where the H_i's are all unbroken the particles associated with nonzero weights acquire a charge and a mass. The elementary spectrum consists of an antisymmetric tensor and four fundamental fields in a hyper multiplet as well as adjoint gauge particles in a vector multiplet. All particles of zero weight are chargeless and massless, and so belongs to a short supermultiplet. These neutral particles are invariant under the $SL(2,Z)$ transformation. We are interested in (charged) BPS spectra, which transforms under the duality, so it is useful to delineate the matter content for each nonzero weight. Breaking up these elementary $Sp(2n)$ multiplets with respects to

the unbroken $U(1)^n$, we find the following distributions of $U(1)^n$ charged $N=2$ supermultiplets,

$$\begin{aligned} \pm\,\gamma_i/2 &\rightarrow \text{4 hypers} \\ \pm\gamma_i &\rightarrow \text{1 vector} \\ \pm\beta_{ij}^{-} &\rightarrow \text{1 vector + 1 hyper} \\ \pm\beta_{ij}^{+} &\rightarrow \text{1 vector + 1 hyper} \end{aligned} \tag{4}$$

Note that each of these charged elementary particles saturate the BPS bound, and thus preserve half of the supersymmetry. The theory possesses the famous $SO(8)$ global symmetry that acts on the fundamental hyper multiplets. Because the fundamental representation is pseudo-real, we can effectively consider the 4 hyper multiplets as being composed of 8 half-hyper multiplets, which transform in the vector representation 8_v of the $SO(8)$. In the following sections, we will show that this elementary spectrum is in fact a part of the full S-dual BPS spectrum.

3 Self-dual BPS Spectra in Generic Vacua

3.1 Classical Solitons and Fermionic Zero Modes

Let the two Higgs expectations in the unitary gauge be denoted by h_i and h'_i. The BPS mass formula for a purely magnetic soliton of charge $\alpha^* = \alpha/\alpha^2$ is

$$M_{\alpha^*} = 4\pi\sqrt{(h\cdot\alpha^*)^2 + (h'\cdot\alpha^*)^2}. \tag{5}$$

Suppose one can write $\alpha^* = \mu^* + \nu^*$ for some roots μ and ν. For generic h and h', the mass formula is not additive and one has [7, 13]

$$M_{\alpha^*} \leq M_{\mu^*} + M_{\nu^*}, \tag{6}$$

where the equality cannot be saturated if

$$(h\cdot\mu^*)(h'\cdot\nu^*) \neq (h'\cdot\mu^*)(h\cdot\nu^*). \tag{7}$$

The condition $(h\cdot\mu^*)(h'\cdot\nu^*) = (h'\cdot\mu^*)(h\cdot\nu^*)$ is satisfied only on a measure zero subset of the vacuum moduli space where h and k are parallel in the (μ,ν) plane, so in a generic vacuum the α^* monopole is classically stable against a decay into a μ^* monopole and a ν^* monopole.

Even in such a generic vacuum, it is still possible to embed the spherically symmetric $SU(2)$ BPS monopole solution to a larger gauge group, along any given root α. Using the $U(1)$ R-symmetry of the $N = 2$ superalgebra, one can rotate the Higgs expectation value $h+ih'$ so that $(h\cdot\alpha^*)+i(h'\cdot\alpha^*)$ is real. Once this is achieved, the embedding of the $SU(2)$ soliton may proceed as if there is only one Higgs field. The embedded subgroup generated by

$$\begin{aligned} t^1(\alpha) &= \frac{1}{\sqrt{2\alpha^2}}\left(E_\alpha + E_{-\alpha}\right), \\ t^2(\alpha) &= -\frac{i}{\sqrt{2\alpha^2}}\left(E_\alpha - E_{-\alpha}\right), \\ t^3(\alpha) &= \alpha^* \cdot H. \end{aligned} \tag{8}$$

will be denote by $SU(2)_\alpha$ in this note. Denoting the $SU(2)$ solution with one Higgs by $A_i^s(r;v)$ and $\Phi^s(r;v)$ with the symmetry breaking scale v, the embedded solution for the root α is

$$\begin{aligned} A_i(r) &= A_i^s(r;h\cdot\alpha)t^s(\alpha), \\ \Phi(r) &= \Phi^s(r;h\cdot\alpha)t^s(\alpha) + (h - h\cdot\alpha^*\,\alpha)\cdot H. \end{aligned} \tag{9}$$

The other Higgs field Φ' is frozen at $h'\cdot H$ once we insure $h'\cdot\alpha = 0$. From this, we can easily see that the magnetic charge of such a soliton is α^*, in accordance with the Dirac quantization condition. The bosonic zero modes around such a soliton must only consist of the 3 translational degrees of freedom and a single $U(1)$ phase, and by the supersymmetry it follows that there are only 2 fermionic zero modes from the Dirac spinor field in the adjoint vector multiplet.

The zero mode analysis of BPS monopoles with one real adjoint Higgs was carried out by E. Weinberg. Adding an extra Higgs field is especially simple in the background of an $SU(2)_\alpha$ embedded monopole. As above, let Φ and Φ' be the two Higgs whose expectation values are h and h', respectively, and we use the global R-symmetry to attain $h'\cdot\alpha = 0$. For the $SU(2)_\alpha$ embedded monopoles, the relevant zero mode equations can be written in the spinorial form as [15, 13],

$$[\gamma^i D^i + \gamma^5\Phi_\alpha]\,\Psi + \{h'\cdot H + \gamma^5(h\cdot\alpha)\,y\}\,\Psi = 0, \tag{10}$$

where the operator inside the bracket is built from the $SU(2)_\alpha$ part of the embedded monopole, and the hypercharge y reflects the discrepancy between the Higgs expectation h and its $SU(2)_\alpha$ part $(h\cdot\alpha^*)\,\alpha$,

$$y = \frac{h\cdot H}{h\cdot\alpha} - \frac{\alpha\cdot H}{\alpha^2}. \tag{11}$$

Clearly, the 4 bosonic and 2 fermionic adjoint zero modes around such an embedded α^* monopole should arise from the $SU(2)_\alpha$ triplet sitting at the weights $\alpha, 0, -\alpha$. But for them, both the mass term $h' \cdot H$ and the hypercharge y vanish. These zero-modes are obtained from the $SU(2)$ zero-modes by a straightforward embedding. Obviously the same is true for zero modes around any embedded monopole of higher magnetic charge $k\alpha^*$.

There could be further fermionic zero modes from hyper multiplets. Consider $\alpha = \beta_{ij}^+ = (e_i + e_j)/\sqrt{2}$. The elementary fields fall into various representations with respect to $SU(2)_{\beta_{ij}^+}$. In particular, there are a pair of Dirac fields in the $SU(2)_{\beta_{ij}^+}$ triplet, sitting at the three weights $\beta_{ij}^+, 0, -\beta_{ij}^+$. One Dirac field belongs to the adjoint vector multiplet while the other belongs to the antisymmetric hyper multiplet. Again by an $SU(2)$ embedding, it is clear that the former contains the two adjoint fermionic zero modes that are supersymmetric partners of the bosonic zero modes. At the same time, the latter also must contain two fermionic zero modes by virtue of having the identical coupling to the $(\beta_{ij}^+)^*$ monopole. Furthermore, since the weights of the antisymmetric multiplet are a subset of the adjoint multiplet, there cannot be any other zero modes from this hyper multiplet. Finally, there are 8 $SU(2)_{\beta_{ij}^+}$ doublet Dirac fields from the 4 fundamental hyper multiplet, sitting at $\gamma_i/2, -\gamma_j/2$ or at $-\gamma_i/2, \gamma_j/2$. No fermionic zero mode results from them, however. If there were such an zero mode, there must also be zero modes from a triplet sitting at $\gamma_i, \beta_{ij}^-, -\gamma_j$ or at $-\gamma_i, -\beta_{ij}^-, \gamma_j$. From the adjoint zero mode counting, we know there is no such zero mode.

Thus we learn that the $(\beta_{ij}^+)^*$ monopole carries four fermionic zero modes in all, two from the adjoint vector multiplet, two more from the antisymmetric hyper multiplet and none from the fundamental hyper multiplets. Exactly the same reasoning goes for $\alpha = \beta_{ij}^- = (e_i - e_j)/\sqrt{2}$, so the $(\beta_{ij}^-)^*$ monopole also carries four such zero modes.

The γ_i^* monopole exhibits a somewhat different behavior. Again, the adjoint vector multiplet contributes a $SU(2)_{\gamma_i}$ triplet Dirac field, sitting at $\gamma_i, 0, -\gamma_i$, which in turn contains all the necessary fermionic partners of the bosonic zero modes. Similarly as above, this implies that there are no fermionic zero modes associated with any other adjoint weights including $\beta_{ij}^\pm$, and from this we deduce that the antisymmetric hyper multiplet contributes no zero modes in the γ_i^* monopole background. Finally, the four fundamental hyper multiplets decompose into four $SU(2)_{\gamma_i}$ doublets,

sitting at $\gamma_i/2, -\gamma_i/2$ and remaining $8(n-1)$ singlets. Each doublet gives rise to a single fermionic zero mode. Thus, the γ_i^* monopole carries six fermionic zero modes in all, two from the adjoint vector multiplet, none from the antisymmetric hyper multiplets, and four from the four fundamental hyper multiplets. Note that for these modes also, the mass term $h' \cdot H$ and the hypercharge y vanish.

Finally note that, for a unit-charged α^* monopole, the fermionic zero modes from an $SU(2)_\alpha$ triplet carry spin 1/2 while those from the doublet are spinless. This is due to the well-known shift of angular momentum in the electric-magnetic bound system. To summarize, the monopole carries four bosonic zero modes in a generic vacuum, and in addition carries the fermionic zero modes as described in Table 1. It is worthwhile to repeat the fact that all of this zero modes are obtained from a simple $SU(2)$ embedding.

monopole charge	matter multiplet	# of zero modes (spin)
$(\beta_{ij}^{\pm})^* = \beta_{ij}^{\pm}$	adjoint vector	2 (1/2)
	antisymmetric hyper	2 (1/2)
	fundamental hypers	0
$\gamma_i^* = \gamma_i/2$	adjoint vector	2 (1/2)
	antisymmetric hyper	0
	fundamental hypers	4 (0)

Table 1: The number of fermionic zero modes from various matter fields. Because the two Higgs are completely misaligned, each monopole carries four bosonic zero modes.

3.2 Dyonic Spectrum

We have proposed in the introduction that our theory possesses an exact $SO(8) \ltimes SL(2,Z)$ duality. If this is indeed the case, the BPS spectrum should have infinite towers of dyons built on every elementary particle listed in Eq. (4). In the case of the completely misaligned Higgs expectation value, we will see now that the semiclassical spectrum can be obtained easily from the known results from the $N=4$ and $N=2$ cases with $SU(2)$ gauge symmetry. In this case, the S-duality passes the test quite easily.

For $(\beta_{ij}^{\pm})^* = \beta_{ij}^{\pm}$, we expect the electric vector meson and the magnetic soliton, or rather their

respective supermultiplets, are dual to each other. The spectrum of elementary electric sector of charge $\beta^{\pm}_{ij}$ is made of a vector and a hyper multiplets, whose state content is identical to a $N = 4$ supermultiplet. Furthermore, since they belong to either the adjoint or the antisymmetric tensor, these multiplets are invariant under the flavor symmetry $SO(8)$.

In the previous section we showed that a single $(\beta^{\pm}_{ij})^*$ monopole has four bosonic zero modes and four complex fermionic zero modes (forming two spin doublets). Although the two of the four fermionic zero modes are from the adjoint Dirac fields and the other two from the antisymmetric tensor, they solve the same $SU(2)_{\beta^{\pm}_{ij}}$ triplet zero mode equation. In effect, it is as if these fermionic zero modes arise from the two adjoint Dirac fields of $N = 4$ $SU(2)$ theory. The upshot is then, the monopole dynamics of k identical $(\beta^{\pm}_{ij})^*$ monopoles of this $N = 2$ system is identical to that of k monopoles in $N = 4$ $SU(2)$ system. The latter is known to have a $SL(2, Z)$-invariant tower of $(n_m = p, n_e = q)$ dyons with all co-prime integer pairs [2, 14]. It follows that there is a similar $SL(2, Z)$-invariant dyonic tower of charges $(p\beta^{\pm}_{ij}, q\beta^{\pm}_{ij})$.

Furthermore, the supermultiplet corresponding to each dyon must be effectively a $N = 4$ vector multiplet, which in turn consists of a $N = 2$ vector and a hyper multiplet. Thus, on each weight $\beta^{\pm}_{ij}$, two (p, q) dyonic towers sit, one in the vector multiplet and the other in the hyper multiplet. This is exactly the BPS spectra one obtains by acting $SL(2, Z)$ on the elementary spectrum in Eq. (4).

As $\gamma^*_i = \gamma_i/2$, a single monopole of magnetic charge γ^*_i should be a dual to quarks of electric charge $\gamma_i/2$. As shown in Table 1, there are four bosonic zero modes, and two fermionic zero modes from the adjoint Dirac spinor, and four fermionic zero modes from the fundamental Dirac spinor. This system is exactly like the monopole system that arises in the $N = 2$ scale-invariant $SU(2)$ model of Seiberg and Witten [3]. Similarly as above, we can pretend that we are considering the multi-monopole dynamics of the scale-invariant Seiberg-Witten model, and the S-duality of γ^*_i dyons simply should follow from the S-duality of the BPS spectrum of the latter model.

The monopole dynamics of the $N = 2$ model is a bit more involved than the $N = 4$ case, and so far one and two monopole dynamics were considered in detail. Dyons with unit magnetic charge

$\gamma_i^* = \gamma_i/2$ must come in four hyper multiplet, or eight half-hyper multiplets, transforming under $SO(8)$ as either 8_c or 8_s. The two fermionic zero modes from the adjoint vector simply fill out a half-hyper multiplet, so one should find the degeneracy factor 8 from the fundamental vector bundle. One would have thought that the 4 fundamental zero modes would generate $2^4 = 16$ degeneracy. The reason why one obtains either 8_c or 8_s, can be found in the nontrivial $O(1)$ bundle [20, 3]. Because the fundamental zero modes acquire a phase of -1 upon a Z_2 gauge transformation, the γ_i^* electric charge can be either even or odd depending on whether even or odd number of the fundamental zero modes are excited. Or equivalently, even (odd) charged dyon can have only even (odd) number of fermions excited. In this way, it was found [3, 4] that the dyons of charge $(\gamma_i^*, 2q\gamma_i^*)$ belong to 8_s of the $SO(8)$ flavor symmetry and those of $(\gamma_i^*, (2q+1)\gamma_i^*)$ belong to 8_c. They are related to the quark states of charge $(0, \gamma_i/2) = (0, \gamma_i^*)$ in 8_v by $SL(2,Z)$ transformations. As was stated above, the duality group is a semidirect product $SO(8) \ltimes SL(2,Z)$ where some $SL(2,Z)$ generators act as a triality.

Two identical monopoles of magnetic charge γ_i^* would have the magnetic charge γ_i. The fermionic zero modes of this magnetic sector would be doubled from those for a single monopole. Their dyonic spectrum is again exactly as in the scale-invariant Seiberg-Witten $SU(2)$ theory. By establishing an index theorem in the $O(2)$ index bundle [20] of the fermion zero modes from the fundamental fermions on the Atiyah-Hitchin space [21], the dyonic spectrum of this sector has been shown to be realized as bound states of right quantum numbers [4]. The dyons of charge $(2\gamma_i^*, (2q+1)\gamma_i^*)$ are in 8 half-hyper multiplets, transforming as 8_v of $SO(8)$, and should be considered a part of the dyonic tower sitting at γ_i^*. The dyons of charge $(2\gamma_i^*, 2q\gamma_i^*) = (\gamma_i, q\gamma_i)$, on the other hand, come in the vector multiplet and invariant under $SO(8)$. They are the dual images of the gauge vector multiplet of electric charge γ_i.

4 Monopole Dynamics with Aligned Higgs

The question of monopole dynamics and the BPS spectrum become far more subtle when the two Higgs expectation values are aligned within the Cartan subalgebra. This is because of the additional bosonic and fermionic zero modes that emerge in such cases. With two Higgs expectation h and

h', we have seen that

$$M_{\mu^*+\nu^*} = M_{\mu^*} + M_{\nu^*}, \tag{12}$$

if and only if $(h\cdot\mu^*)(h'\cdot\nu^*) = (h'\cdot\mu^*)(h\cdot\nu^*)$. Thus if the two Higgs are aligned in the (μ,ν) plane, a monopole of charge $\mu^*+\nu^*$ can be split into a pair of monopoles of charges μ^* and ν^* without costing any energy. This translates into additional massless degrees of freedom in the low energy monopole dynamics, and the moduli space dynamics becomes more involved.

While one may expect that the S-duality of the BPS spectrum in generic vacuum will continue to hold in such exceptional cases, the $N=2$ supersymmetric theories are known to exhibit discontinuity in spectra as one moves around the vacuum moduli space. An explicit check is necessary. If the S-duality holds everywhere, this would imply that certain sets of monopoles must combine quantum mechanically to give rise to supersymmetric quantum bound states of specific nature. While we could not construct all such bound states necessary, there turned out to be many cases where the existence of the bound states and even their precise forms can be inferred from the known results in $N=4$ Yang-Mills theory. We start the section by revisiting the matter of zero modes, and attack the problems of various dyon spectra, weight by weight. For notational simplicity, we will consider the case where h is parallel to h' in the entire Cartan subalgebra.

4.1 Zero Modes Revisited

When the two Higgs expectations are aligned, one can rotate $h+ih'$ by the global $U(1)$ R-symmetry so that we may effectively set one of the Higgs expectation, say h', to zero and subsequently let the corresponding Higgs field Φ' vanish as well. The spinorial zero-mode equation of Section 3.1 is then reduced to

$$[\gamma^i D^i + \gamma^5 \Phi_\alpha]\,\Psi + \gamma^5 (h\cdot\alpha)\, y\,\Psi = 0. \tag{13}$$

Recall that the hypercharge y reflects the discrepancy between the Higgs expectation h and its $SU(2)_\alpha$ part $(h\cdot\alpha^*)\,\alpha$. The number of zero modes is determined by the $SU(2)_\alpha$ representation and the hypercharge, and Table 2 summarizes the results obtained by Weinberg [15].

$SU(2)_\alpha$ representation	hypercharge	# of zero modes
singlet	any y	0
doublet	$1/2 > \|y\|$	1
	$\|y\| \geq 1/2$	0
triplet	$1 > \|y\|$	2
	$\|y\| \geq 1$	0

Table 2: The number of Dirac zero modes for an $SU(2)$ embedded, spherically symmetric monopole of charge α^*. The $SU(2)_\alpha$ representation is that of the Dirac spinor.

Now suppose we arranged the Cartan generators such that the Higgs expectation h satisfies the following inequality,

$$0 < h_1 < h_2 < \cdots < h_{n-1} < h_n. \tag{14}$$

Then a set of n roots $\{\mu_i\} = \{\gamma_1, \beta^-_{21}, \ldots, \beta^-_{n,n-1}\}$ has a natural interpretation as the positive simple roots. Consequently, a μ^*_i monopole has exactly 4 bosonic zero modes so that one can interpret it as a fundamental soliton. For monopoles of total charge $k\mu^*_i$ with any positive integer k, the zero mode counting remains unchanged from the previous section. Others will acquire more bosonic and fermionic zero modes, and we may proceed with the counting with the help of Table 2, keeping in mind that the number of the zero modes is additive in this case of completely aligned Higgs. Table 3 summarizes the number of fermionic zero modes for monopoles of various charges, assuming that h is parallel to h' in the unitary gauge. Some of the zero modes will be lifted if h is partially misaligned with h'.

monopole charge	matter multiplet	# of zero modes	bundle structure
$k(\beta^-_{ij})^* = k\beta^-_{ij}$	adjoint	$(2i-2j)k$	$Sp(2ik-2jk-2)$
	antisymmetric	$(2i-2j)k$	$Sp(2ik-2jk-2)$
	fundamental	0	$\cdot$
$k\gamma^*_i = k\gamma_i/2$	adjoint	$2ik$	$Sp(2ik-2)$
	antisymmetric	$(2i-2)k$	$Sp(2ik-2k)$
	fundamental	$4\times k$	$O(k)$
$k(\beta^+_{ij})^* = k\beta^+_{ij}$	adjoint	$(2i+2j)k$	$Sp(2ik+2jk-2)$
	antisymmetric	$(2i+2j-4)k$	$Sp(2ik+2jk-4k)$
	fundamental	$4\times 2k$	$O(2k)$

Table 3: The number of fermionic zero modes from various matter sectors. The number of bosonic zero modes is twice that of the fermionic adjoint zero modes. The last column

shows the structure group of the vector bundle spanned by these zero modes. See Section 4.4 for detailed explanations.

4.2 The μ_i^* Dyons

As observed above, the n roots $\{\mu_i\} = \{\gamma_1, \beta_{21}^-, \ldots, \beta_{n,n-1}^-\}$ correspond to the n fundamental monopoles. The simplest dyon spectra one can address are those of magnetic charges $k\mu_i^*$. But for these fundamental dyons, the zero mode counting and the moduli space dynamics are identical to that of the previous section, and we need not ask anything new. The S-duality of μ_i^* dyons (and $2\gamma_1^*$ dyons as well) in the aligned Higgs case follows from that of the misaligned Higgs case.

4.3 The $(\beta_{ij}^-)^*$ Dyons

Note the following very useful fact about the $(\beta_{ij}^-)^* = \beta_{ij}^-$ monopoles. First, the roots $\{\beta_{ij}^-, -\beta_{ij}^-\}$, together with the $(H_i - H_{i+1})$'s, span an $SU(n)$ subgroup of the $Sp(2n)$. Any BPS monopole of charge $(\beta_{ij}^-)^*$ can be subsequently regarded as an embedding of a $SU(n)$ monopole of the same charge to the larger $Sp(2n)$ gauge group, again up to a uniform part of the Higgs field proportional to $\sum_i H_i$. Then, the adjoint representation of $Sp(2n)$ is decomposed into an adjoint, a symmetric tensor and its conjugate, as well as a singlet of the said $SU(n)$ subgroup, while the antisymmetric tensor of $Sp(2n)$ consists of an adjoint, an antisymmetric tensor and its conjugate of the same $SU(n)$. But in fact, we know from the zero-mode counting of the $SU(n)$ monopole that all $2i-2j$ zero modes arise from the adjoint of the $SU(n)$ alone, as a $(\beta_{ij}^-)^*$ monopole is made of $i-j$ distinct fundamental monopoles. Consequently, all the antisymmetric zero modes must fall into the adjoint of this $SU(n)$ as well.

Since the $\sum_i H_i$ part of the Cartan algebra commutes with the embedded $SU(n)$, this implies that the fermionic zero modes from the vector multiplet solve exactly the same equations as those from the antisymmetric hyper multiplet do. Effectively, the number of the fermionic zero modes from the vector multiplet is doubled. But we are already familiar with such a multi-monopole dynamics. It is a multi-monopole dynamics obtained from $N=4$ $SU(n)$ Yang-Mills theory (with

all six Higgs aligned), which has been studied thoroughly by the authors as well as others [18, 19]. By now it is widely believed and comprehensively tested that such an $N = 4$ theory possesses fully S-dual BPS spectrum at the semiclassical level. In other words, the well-known S-duality of $N = 4$ $SU(n)$ Yang-Mills theory automatically implies that there is an $SL(2, Z)$-invariant dyonic tower of charges $(p\beta^-_{ij}, q\beta^-_{ij})$ for all co-prime integer pair (p, q). Explicit forms of the bound states is known for $p = 1$ [18, 19].

Finally, because the monopole dynamics is effectively that of the $N = 4$ Yang-Mills theory, the bound states one gets are in the $N = 4$ vector multiplet. An $N = 4$ vector multiplet is in turn composed of one $N = 2$ vector multiplet and one $N = 2$ hyper multiplet, so we know that on each weight $\pm(\beta^-_{ij})^* = \pm\beta^-_{ij}$ there are a pair of (p, q) dyonic towers, one in vector and the other in hyper. This is exactly the spectrum one would obtain by performing $SL(2, Z)$ transformation on the elementary BPS spectra on $\pm\beta^-_{ij}$ delineated in Eq. (4).

4.4 The Index Bundle

Before proceeding further, we need to elaborate on some basic facts about the moduli space dynamics. The zero modes can be thought of deviations from the classical monopole background. For bosonic zero modes from the vector multiplet, this is especially clear since they literally encode small deformation of the classical BPS monopole solution. The fermionic zero modes deform the state at quantum level, on the other hand, by supplying an additional set of fermionic harmonic oscillators to the moduli space quantum mechanics. One consequence is that the wavefunction becomes a multi-component variety, each component being characterized by which subset of these oscillators is excited.

The extra interactions introduced through such fermionic excitations arise naturally from the field theory as follows. Consider a Dirac Lagrangian,

$$\mathcal{L} = \int d^4x \; \{i\bar{\Psi}\gamma^\mu D_\mu \Psi + \cdots\}\,. \tag{15}$$

When a background Yang-Mills field carries m number of Ψ zero modes, say Ψ_p, $p = 1, ..., m$, their contribution to the moduli space dynamics is found by expanding $\Psi = \sum \eta_p(t)\Psi_p$ with the

time-dependent, complex Grassmannian, collective coordinates η_p, and inserting it back to the Lagrangian. Only the time derivative piece survives, and one finds [20, 22]

$$\mathcal{L} \quad \rightarrow \quad \int dt \, \left\{ i\tilde{\eta}_p \frac{d}{dt}\eta_p + i\tilde{\eta}_p\eta_q\mathcal{A}_{apq}\frac{dz_a}{dt} + \cdots \right\}, \tag{16}$$

where the tilde denotes the complex conjugation. The z_a's are the bosonic collective coordinates that parameterize the background Yang-Mills field, and the "connection" $\mathcal{A}$ is given by

$$\mathcal{A}_{apq} = \int dx^3 \; \Psi_p^\dagger \delta_a \Psi_q. \tag{17}$$

This connection is easily seen to be anti-hermitian, $\mathcal{A}_{apq} = -\tilde{\mathcal{A}}_{aqp}$. In this sense, the m fermionic zero modes can be regarded as spanning a $U(m)$ vector bundle on the moduli space.

When the gauge group representation of Ψ is real or pseudo-real, however, there is a further constraint on $\mathcal{A}_{apq}$. The Dirac operator is then equivalent to its charge conjugate up to a redefinition of the spinor,

$$\Psi \quad \Rightarrow \quad \Psi_c = S \otimes V \, \tilde{\Psi}. \tag{18}$$

The unitary matrix S acts on the group indices and is either symmetric if the group representation is real or antisymmetric if the group representation is pseudo-real. Alternatively, $S\tilde{S} = \pm 1$ for real and pseudo-real representations, respectively. The unitary matrix $V = -V^T$ is the charge conjugation matrix for the $SO(3,1)$ Dirac spinor. This means in particular that if Ψ_p is a zero mode, so is $(\Psi_p)_c$. Thus, the two must be related by yet another unitary transformation $\mathcal{C}$,

$$(\Psi_p)_c = \mathcal{C}_{pq}\Psi_q, \tag{19}$$

where $\mathcal{C}\tilde{\mathcal{C}} = \mp 1$, or equivalently $\mathcal{C} = \mp\mathcal{C}^T$, for $S\tilde{S} = \pm 1$. A unitary redefinition of the basis $\Psi_p \Rightarrow \Psi_q\tilde{U}_{qp}$ rotates $\mathcal{C}$ to $U^T\mathcal{C}U$, which clearly preserves $\mathcal{C}\tilde{\mathcal{C}}$. Finally, taking the determinant of $\mathcal{C}\tilde{\mathcal{C}}$, we learn that the number of zero modes must be even, $m = 2k$, whenever $\mathcal{C}\tilde{\mathcal{C}} = -1$.

Using this charge conjugation twice on $\mathcal{A}_a$, we find the following reality constraint,

$$\mathcal{A}_a\mathcal{C} + \mathcal{C}\mathcal{A}_a^T = 0. \tag{20}$$

The unitarity guarantees that $\mathcal{C}$ can be brought into a (skew-)diagonalized form by an orthogonal U, and then $\mathcal{C}\tilde{\mathcal{C}} = \mp 1$ tells us that the (skew-)eigenvalues are pure phases. These phases are easily

absorbed by the zero modes themselves. The canonical form of $\mathcal{C}$ is then

$$\mathcal{C} = 1 \otimes i\tau_2 \quad \text{if the group representation is real,} \tag{21}$$

$$\mathcal{C} = 1 \quad \text{if the group representation is pseudo-real.} \tag{22}$$

We can easily recognize these as the invariant bilinear forms of $Sp(m = 2k)$ and $O(m)$, respectively. Thus, when the fermions are in real and pseudo-real representations of the gauge group, the structure group of the index bundle reduces to $Sp(m = 2k)$ and $O(m)$, respectively.

The adjoint representation of any gauge group is real, so we should find a $Sp(2k)$ bundle to emerge from the $2k$ fermionic zero modes of the vector multiplet. Because of the $N = 2$ supersymmetry, on the other hand, this bundle is equivalent to the co-tangent bundle of the $4k$-dimensional monopole moduli space, so the latter must have the structure group $Sp(2k)$. A $4k$-dimensional manifold with a structure group $Sp(2k)$ is by definition hyperkähler, and thus we have rediscovered the well-known fact that the moduli space is naturally hyperkähler. In fact, the actual structure group of the moduli space of $4k$ dimensions is somewhat smaller, $Sp(2k - 2)$, because the 4-dimensional, flat center-of-mass part factors out in the metric locally. Thus, by supersymmetry, $2k$ fermionic zero modes of the vector multiplet is decomposed as $2 + (2k - 2)$, where the two associated with the center-of-mass coordinates span a flat bundle, while the remaining $(2k - 2)$ span a $Sp(2k - 2)$ vector bundle over the relative moduli space.

The fundamental representation of $Sp(2n)$ is pseudo-real, which tells us that the resulting index bundle is a $O(m)$ vector bundle. This generalizes the observation by Manton and Schroers that the zero mode bundle from a fundamental representation of $SU(2) = Sp(2)$ is a real vector bundle [20]. This fact was a crucial ingredient in studying the S-dual spectrum of scale-invariant $N = 2$ $SU(2)$ model.

The most interesting aspect of the Yang-Mills theory we are considering in this note, is the presence of the antisymmetric matter. The group representation is also real, so $2k'$ fermionic zero modes span a $Sp(2k')$ vector bundle in general. Note that the case of $(\beta^-_{ij})^*$ monopoles in Section 4.3 is an exceptional case; instead of spanning $Sp(2i - 2j)$ as the structure group, as the general argument would imply, the $2k' = 2i - 2j$, zero modes turned out to span a $Sp(2i - 2j - 2)$ bundle. This happened because the antisymmetric zero modes are actually identical to the adjoint ones.

Recall that the structure group of the adjoint index bundle with $2k = 2i - 2j$ zero modes is $Sp(2i - 2j - 2)$ because of the moduli space decomposition.

4.5 The γ_2^* Dyons

The next nontrivial problem is that of $\gamma_2^* = \gamma_2/2$ dyons. The purely magnetic state must be realized as a threshold bound state of a γ_1^* monopole and a $(\beta_{21}^-)^*$ monopole. Is the monopole dynamics again effectively that of $N = 4$ Yang-Mills as in the $(\beta_{ij}^-)^*$ dyons? Not quite so. The γ_2^* monopole carries 4 adjoint zero modes, 2 antisymmetric tensor modes and 4×1 fundamental zero modes. There is no obvious doubling of the adjoint fermionic modes. Nevertheless, the bound state problem can be mapped to an $N = 4$ problem as follows.

Although the adjoint multiplet has twice the number of zero mode as the antisymmetric one, their structure groups are both $Sp(2)$, because of the usual decomposition of the moduli space $\mathcal{M}$ [18],

$$\mathcal{M} = R^3 \times \frac{R^1 \times \mathcal{M}_0}{Z}. \tag{23}$$

Of the four adjoint modes, two are supersymmetric partners of the center-of-mass coordinates parameterizing $R^3 \times R^1$. These two do not participate in the interaction and simply provide the $N = 2$ BPS supermultiplet structure to the wavefunction. As far as the mutual monopole interaction goes, we have a pair of $Sp(2)$ vector bundles on $\mathcal{M}_0$, one from the adjoint vector multiplet and the other from the antisymmetric hyper multiplet. For a pair of distinct fundamental monopoles, the relative moduli space $\mathcal{M}_0$ is always the Taub-NUT manifold [23, 18], the metric on which can be written as

$$\mathcal{G} = \left(1 + \frac{1}{r}\right) [dr^2 + r^2\sigma_1^2 + r^2\sigma_2^2] + \frac{1}{1 + 1/r}\,\sigma_3^2 \tag{24}$$

up to an overall constant. The 3 one-forms $\sigma_i/2$ are an orthonormal basis on a unit three sphere, which is invariant under the spatial rotation $SU(2)$. In terms of the Euler angles θ, φ, ψ, they are

$$\begin{aligned} \sigma_1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\varphi, \\ \sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta \varphi, \\ \sigma_3 &= d\psi + \cos\theta d\varphi. \end{aligned} \tag{25}$$

Because the Taub-NUT manifold is simply connected, a nontrivial twisting under a $O(1)$, if any, can occur only through the modding by the integer group, and fundamental zero modes decouple from the relative dynamics also. Thus, only the adjoint and antisymmetric zero-modes enter the relative moduli space dynamics.

Let (η^1, η^2) be the two complex collective coordinates associated with the anti-symmetric zero modes. Also let ψ^a be the four real, supersymmetric partners of the bosonic collective coordinates z^a. The relative part of the moduli space dynamics is governed by the supersymmetric Lagrangian,

$$\int dt\, \left\{ \frac{1}{2}\mathcal{G}_{ab}\dot{z}^a\dot{z}^b + \frac{i}{2}\psi^a D_t\psi^a + i\eta^\dagger \mathcal{D}_t \eta + \frac{i}{2}\eta^\dagger [\mathcal{F}_{ab}\psi^a\psi^b]\eta \right\}, \tag{26}$$

where the two covariant derivatives

$$\begin{aligned} D_t\psi^a &= \partial_t\psi^a + \dot{z}^c w_{abc}\psi^b, \\ \mathcal{D}_t\eta^p &= \partial_t\eta^p + \dot{z}^c \mathcal{A}_{cpq}\eta^q, \end{aligned} \tag{27}$$

are defined with respect to the connections of the co-tangent bundle and the index bundle of the anti-symmetric zero modes, respectively. $\mathcal{F}$ is the curvature of the latter, $d\mathcal{A}+\mathcal{A}^2$. The quantization of the ψ^a's dictates that the wavefunction should be a Dirac spinor on $\mathcal{M}_0$ [4], while η quantization leads either to one of two $Sp(2)$ singlets or to a $Sp(2)$ doublet. More explicitly, the canonical quantization of η leads to a pair of fermionic harmonic oscillators,

$$\{\eta_i, \tilde{\eta}_j\} = \delta_{ij}. \tag{28}$$

The vacuum $|0\rangle$ is defined to be annihilated by the η_i's, then, the four states are $|0\rangle$, $\tilde{\eta}_1\tilde{\eta}_2|0\rangle$, $\tilde{\eta}_1|0\rangle$, and $\tilde{\eta}_2|0\rangle$. The last two form a doublet, while the first two are singlets. The Hamiltonian is the square of a supersymmetry generator $\mathcal{Q}$, which is in effect a Dirac operator on $\mathcal{M}_0$ [22, 4].

In order to find the bound states in the relative moduli space, we need to know the $Sp(2)$ connection $\mathcal{A}$ over $\mathcal{M}_0$. The moduli space dynamics inherits the symmetries of the underlying monopole dynamics, namely a rotational $SU(2)$ and a gauge $U(1)$ [4, 18], and $\mathcal{F}$ must respect them as well. Furthermore, a theorem due to Hitchin says the curvature must be anti-self-dual in the space of differential forms [20], so the $Sp(2)$ connection $\mathcal{A}$ must be that of $SU(2)\times U(1)$-invariant anti-instanton(s) on $\mathcal{M}_0$. It turns out that there is a one-parameter family of such $Sp(2)$

connections, up to gauge transformations,

$$\mathcal{A}_a dz^a = \left[1 - \frac{1}{1+r/R}\right]\left\{\sigma_1 T^1 + \sigma_2 T^2\right\} + \left[1 - \frac{1}{(1+r/R)(1+r)}\right]\sigma_3 T^3, \tag{29}$$

where $[T^j, T^k] = -\epsilon_{ijk}T^i$ are the anti-hermitian $Sp(2) = SU(2)$ generators, and the size parameter R is positive. This connection is invariant under the $U(1)$ isometry which rotates the (σ_1, σ_2) pair, as long as one rotates the (T^1, T^2) pair simultaneously.

In fact, the connection w of the Taub-NUT manifold $\mathcal{M}_0$ can also be regarded as a special case of this one parameter family. This has to be the case, since w is also effectively an $Sp(2)$ connection whose curvature is anti-self-dual and invariant under the isometry of $\mathcal{M}_0$. The self-dual part $w^{(+)}$ is curvature free, and can be transformed away in one of the two $SU(2)$'s in $SO(4) = SU(2) \oplus SU(2)$. In contrast, the anti-self-dual part $w^{(-)}$ takes values in the other $SU(2)$, and must be interpretable also as an $SU(2) = Sp(2)$ anti-instanton. A direct computation shows that the $w^{(-)}$ coincides with $\mathcal{A}_a dz^a$ with $R = 1$, after an appropriate reinterpretation of T^i as Euclidean Lorentz transformation generators. Alternatively, starting with an $Sp(2) = SU(2)$ bundle with the connection $w^{(-)}$, we can build an $SO(4)$ vector bundle by swapping the two complex coordinates similar to the η's for four real ones. By gauge rotating the curvature free part of the $SO(4)$, which mixes holomorphic and anti-holomorphic coordinates, we may induce a curvature free $w^{(+)}$, and once this is done the result is simply the co-tangent bundle on the Taub-NUT. In fact, this is one way to recover the adjoint part of the Lagrangian starting with two fermionic zero modes from the vector multiplet.

The Pontryagin number of $\mathcal{A}$ is easily seen to be -1, and this is an anti-instanton of arbitrary "size" R. Can the anti-instanton be of an infinite size? Unless $R = \infty$, the asymptotic form of $\mathcal{A}$ on the surface S^3 spanned by the σ_i's as $r \to \infty$ is pure gauge of unit winding number, and can be removed by a large gauge transformation on S^3. On the other hand, if $R = \infty$, the anti-instanton becomes infinitely large and $\mathcal{A}$ degenerates to a $U(1)$ connection $r/(1+r)\sigma_3 T^3$ whose field strength $\mathcal{F} \sim -T^3 \sin\theta\, d\theta\, d\phi$ is not a pure gauge on the asymptotic S^3. A useful thing to note is that, when $r \to \infty$, zero modes from the anti-symmetric hyper multiplet are localized at the $(\beta^-_{21})^*$ monopole and in fact solve the same equation as the adjoint zero-modes around the $(\beta^-_{21})^*$ monopole, to the leading order in $1/r$. This, together with the above identification of w with $\mathcal{A}$ at $R = 1 < \infty$, implies that the connection of the index bundle of η must also be pure gauge on the asymptotic S^3. Thus, we learn that the "size" of the anti-instanton must be finite, $R < \infty$, and the index bundle

does not degenerate to a $U(1)$ type.

An independent way to see $R < \infty$ is to consider the spin of the zero modes. Consider the Noether angular momentum obtained from the Lagrangian (26), neglecting the ψ^a modes, by using the rotation Killing vectors,

$$\begin{aligned} L_x &= -\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\sin\theta}(\frac{\partial}{\partial\psi} - \cos\theta\frac{\partial}{\partial\varphi}), \\ L_y &= \cos\varphi\frac{\partial}{\partial\theta} + \frac{\sin\varphi}{\sin\theta}(\frac{\partial}{\partial\psi} - \cos\theta\frac{\partial}{\partial\varphi}), \\ L_z &= \frac{\partial}{\partial\varphi}. \end{aligned} \tag{30}$$

The anomalous contribution from η, in addition to the familiar bosonic part $\mathbf{J}_B = \mathbf{r}\times\dot{\mathbf{r}} + q_B\hat{\mathbf{r}}$ with $q_B = (\sigma_3/dt)/(1+1/r)$, is given by

$$\Delta J_i = q_\eta \hat{r}_i + S_i \tag{31}$$

where $q_\eta = -i\tilde{\eta}T^3\eta/[(1+r/R)(1+r)]$ is an $U(1)$ electric charge from η and, at large r with any finite fixed R, S_i converges to

$$S_i^\infty = -i\tilde{\eta}U^\dagger T^i U\eta, \tag{32}$$

with a large gauge transformation U on S^3, which removes the asymptotic value of the connection $\mathcal{A}$. Upon the quantization of η, we find $[S_i^\infty, S_j^\infty] = i\epsilon_{ijk}S_k^\infty$. Of the four states generated by η quantization, the $Sp(2)$ doublet $\tilde{\eta}_i|0\rangle$ contributes a quantized spin 1/2, while the singlets $|0\rangle$ and $\tilde{\eta}_1\tilde{\eta}_2|0\rangle$ contribute no spin. In such a limit of infinite separation, the two anti-symmetric zero modes arise from a triplet of $SU(2)_{\beta_{21}^-}$ and form a spin doublet, so this behavior of angular momentum is exactly what one should have expected. On the other hand, if $R = \infty$, the anomalous contribution would be of the form,

$$\Delta J_i = q'_\eta \hat{r}_i \tag{33}$$

for a different $U(1)$ charge q'_η from η, and devoid of such a spin contribution. (The difference $q_\eta - q'_\eta$ is independently conserved when $R = \infty$.) This provides a second evidence that R should be finite.

Now we are ready to find the threshold bound state of purely magnetic charge γ_2^*. Suppose for the sake of argument that $R = 1$ and $\mathcal{A} = w^{(-)}$. The index bundle from the antisymmetric hyper multiplet is then equivalent to the co-tangent bundle on the Taub-NUT, so effectively we have doubled the adjoint fermionic zero modes on the relative moduli space. The four real degrees of

freedom in η would behave exactly as the ψ^a's, after an appropriate redefinition of the coordinates, and the dynamics is again that of a pair of distinct fundamental monopoles from $N = 4$ Yang-Mills theory. For instance, the curvature term in the effective Lagrangian will turn into that of $N = 4$ system involving the Riemann curvature of $\mathcal{M}_0$. Such an $N = 4$ system has been studied previously and a unique bound state is known to arise [18].[3]

What happens if $R \neq 1$? Recall that the bound state in question must be and is annihilated by the supersymmetry generator $\mathcal{Q}$, and thus contributes to the Witten index. So the Witten index with an L^2 condition is equal to one. But an index is a topological quantity that cannot change with small perturbation. As long as R remains positive and finite, nothing drastic happens, and the Witten index remains 1; there must be at least one bosonic bound state.

Furthermore, a careful look at the supersymmetry generator, or equivalently a Dirac operator on the Taub-NUT, reveals that in fact there is no extra bound state. The wavefunction is a Dirac spinor on the Taub-NUT space and could be either a singlet or a doublet under the $Sp(2)$. Using the Weitzenbock formula, the square of the Dirac operator (which is the Hamiltonian) can be written as

$$\mathcal{Q}^2 = -\nabla\nabla + \frac{\kappa}{4} + i\mathcal{F}_{ab}\Gamma^{ab}, \tag{34}$$

where κ is the scalar curvature of Taub-NUT, and $2\Gamma^{ab} = [\Gamma^a, \Gamma^b]$ with the $SO(4)$ Dirac matrices Γ^a. The hyperkähler property ensures $\kappa = 0$, while $\mathcal{F}_{ab}\Gamma^{ab}$ is null except on anti-chiral part of the spinor in a $Sp(2)$ doublet. So a nontrivial solution to the Dirac equation can arise only from anti-chiral spinor in $Sp(2)$ doublet. On the other hand, the Witten index in question counts the number of bound states in the form of anti-chiral spinor in a $Sp(2)$ doublet or chiral singlet, minus the number of those in the form of a chiral $Sp(2)$ doublet or an anti-chiral singlet. Since only one of the four sectors can contribute to the Witten index, the index actually counts the total number of the normalizable ground states.

Finally, the supersymmetric wavefunction found at $R = 1$ is invariant under the $U(1)$ gauge

[3] In the $N = 4$ $Sp(2n)$ theory, a vector multiplet would be generated from this bound state in $\mathcal{M}_0$ and becomes dual to certain elementary gauge-particles in the GNO-dual [24] $SO(2n + 1)$ theory. However, the center-of-mass part of the index bundles we have differs from the one found in $N = 4$ case, and because of this, the bound state generates eight half-hyper multiplets, dual to quarks, instead. This explains why the gauge group is not mapped to its GNO-dual group but rather to itself.

isometry of the Taub-NUT [18] and because of this, has no relative electric charge. The electric charge is quantized and cannot change under a continuous shift of a parameter, so whatever the actual value of R is, the bound state in $\mathcal{M}_0$ is purely magnetic of charge γ_2^*. Once we know this, the rest of the quantization may proceed just as in the $SU(2)$ case [4].[4]

One important difference from the $(\beta_{ij}^-)^*$ dyon case is the fermionic zero modes associated with the center-of-mass part of the moduli space. Instead of two identical copies of a pair of adjoint zero modes from the vector multiplet, as would be appropriate for the actual $N = 4$ system, we have one such pair and four copies of a $O(1)$ bundle from the four fundamental hyper multiplets. The four copies of $O(1)$ bundle play the same role as in the $SU(2)$ case, and generates $8_s + 8_c$ towers of $(\gamma_2^*, q\gamma_2^*)$ dyons where 8_s has even q and 8_c has odd q. The remaining two adjoint zero modes associated with the center-of-mass part carry spin $\pm 1/2$, and generates a short supermultiplet structure of degeneracy 4, a half-hyper multiplet. For each electric charge q, we conclude, there are $1/2 \times 8 = 4$ hyper multiplets of charge $(\gamma_2^*, q\gamma_2^*)$, which clearly constitute part of the 4 $SL(2,Z)$-invariant (p,q) dyonic towers in hyper multiplet that should be sitting at $\gamma_2^* = \gamma_2/2$.

4.6 The γ_i^* Dyons for $i \geq 3$

A magnetic charge of γ_i^* is a sum of the following fundamental charges,

$$\gamma_i^* = \gamma_1^* + (\beta_{21}^-)^* + (\beta_{32}^-)^* + \cdots + (\beta_{i,i-1}^-)^* = \mu_1^* + \mu_2^* + \cdots + \mu_i^*. \tag{35}$$

The appropriate multi-monopole moduli space is known [16], so it only remains to see what the hyper-multiplet zero modes do. There are still 4 fundamental modes forming four copies of a $O(1)$ bundle, but thanks to the trivial topology of the relative moduli space, these do not enter the relative dynamics. The antisymmetric zero modes and the relative part of the adjoint zero modes each span a $Sp(2i-2)$ vector bundle, and the curvatures of the two bundles enjoy the same symmetry properties. If the two bundles are actually equivalent, the relative moduli space dynamics again reduces to $N = 4$ dynamics and the S-duality of $N = 4$ Yang-Mills system implies that we recover exactly one bound state [19]. Another possibility is that one index bundle are smooth nonsingular deformations of the other, in which case we still have the Witten index equal

[4] While decomposition of the monopole moduli space differs from the $N = 2$ $SU(2)$ case in Ref. [4], this does not affect the results because the bound state in $\mathcal{M}_0$ is invariant under the discrete modding by Z.

to 1. By quantizing the center-of-mass part of the moduli space, one would obtain $(\gamma_i^*, q\gamma_i^*)$ dyons where even q's are in 8_s and each q's are in 8_c under $SO(8)$. This is clearly consistent with the $SL(2,Z)$-invariant (p,q) dyonic towers on γ_i^*.

Thus, the interesting question is whether one can actually show that the two bundles are related by a continuous small deformation, or even equivalent. At the moment, we have no compelling argument why that should be so, and clearly more study is necessary to address this problem.

4.7 Others

So far, we addressed the $(1,q)$ part of the (p,q) dyonic towers located at $(\beta_{ij}^-)^* = \beta_{ij}^-$ and $\gamma_i^* = \gamma_i/2$. Because of the aligned nature of the Higgs expectations, these states appear already as multi-monopole bound states. Due to the lack of knowledge on the general multi-monopole dynamics with many identical monopoles, it appears that the semiclassical approach is impractical in finding the full (p,q) tower of the dyons on these weights.

Also, there are remaining weights, $(\beta_{ij}^+)^* = \beta_{ij}^+$ and $2\gamma_i^* = \gamma_i$. In order for the S-duality to hold, there must be a vector and a hyper multiplet for each charge $(p\beta_{ij}^+, q\beta_{ij}^+)$ for all co-prime integer pairs (p,q), and similarly a $(p\gamma_i, q\gamma_i)$ tower of vector multiplets. (The existence of the right $2\gamma_1^* = \gamma_1$ dyonic tower actually follows from the discussion in Section 3.2, thanks to the fact γ_1 is one of the preferred simple root.) The quantization of the moduli space dynamics for these is fairly involved. For one thing, we need certain multi-monopole moduli spaces of so far unknown geometry. The amount of evidences we collected in the previous two sections, however, are rather overwhelming, and we have no reason to believe that the S-duality should fail for these special cases.

5 Conclusion

We have presented several strong evidences for the $SL(2,Z) \ltimes SO(8)$ duality in the scale-invariant $N=2$ $Sp(2n)$ theory with one antisymmetric and four fundamental hyper multiplets. These

evidences are found in the semiclassical dyonic spectrum of the theory that matches the expected duality-invariant one. When the Higgs vacuum expectation values are not aligned, the right dyonic spectrum for arbitrary n is shown to arise as a consequence of well-established S-dualities of the $N = 4$ $SU(2)$ theory and of the scale-invariant $N = 2$ $SU(2)$ theory. When the two expectation values are parallel, the monopole dynamics are more involved. We identified the index bundle on the monopole moduli space, generated by the zero modes from the hyper multiplets, and quantized the resulting moduli space dynamics. Some of the multi-monopole bound states and the dyonic excitations are constructed, and the result is fully consistent with the proposed S-duality.

In this note, we ignored the four Higgs from the antisymmetric tensor hyper multiplet. In the F-theory picture, where the adjoint Higgs encodes the D3-brane positions in the two compact directions, antisymmetric Higgs encode the four position coordinates of the n D3-branes along the noncompact direction. They will contributes to the masses of dyons built on $\beta_{ij}^{\pm}$ weights. But we believe this slight complication does not alter the conclusion we drew above.

There are some open questions for the case of the aligned Higgs. The most obvious ones concern the part of the spectra which we did not test explicitly, such as $(\beta_{ij}^{+})^*$ dyons, for the lack of knowledge about the moduli space geometry. On the other hand, there are more accessible problems left unanswered in this note. For example, we did not determine the precise connection on the $Sp(2)$ index bundle from the hyper multiplet in Section 4.5 and still were able to deduce the right spectrum. More close study of such nonabelian bundles on the moduli space should be rewarding in taking the next step and uncovering dyons of higher magnetic charges.

We have started with the F-theory viewpoint of the Yang-Mills system as the motivation. It gave rise to a particular generalization of the scale-invariant $SU(2)$ model, and the U-duality of the type IIB theory in turn strongly suggested that such a generalization would lead to exactly S-dual $N = 2$ theories, which we tested explicitly in the semiclassical framework. It makes us wonder how far one can go in realizing $N = 2$ and maybe even $N = 1$ supersymmetric Yang-Mills systems as the world-volume theory of D-branes. Recently, there has been a flurry of research activities in this regard, reproducing known, sometimes very nontrivial, field theory results or even leading to completely new insights and systems[26]. Within the present context of searching for theories with exact S-duality, it could be also worthwhile to consider other string vacua and D-branes configurations and

see what conclusions one can draw about resulting supersymmetric systems.

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